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A Fundamental Solution of the Parabolic Equation on Hilbert Space*

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I. INTRODUCTION

In this paper we investigate existence and regularity of solutions of a second order parabolic equation of the form $\partial u/\partial t = Lu$, where L is a second order differential operator, with variable coefficients, involving derivatives in an infinite number of orthogonal directions.

Gross [9] has developed the theory of the Laplacian on a real separable Hilbert space H. There the Laplacian of a real-valued function f on H is defined as the trace of the second Frechet derivative f'' of f, when the latter exists. This reduces to the usual definition when H is finite dimensional.

In order to investigate the Dirichlet problem for the equation $\Delta u = f$ together with the regularity properties of the solutions, it is necessary to embed H in a Banach space B (e.g. Wiener space) large enough to carry a countably additive extension of Gauss measure n_t on H. This extension is denoted by p_t, and is called Wiener measure on B with variance parameter t. Under suitable hypotheses on B, Gross has shown that for a bounded Lip 1 function f on B, $p_t f(x) \equiv \int_B f(y) p_t(x, dy)$ has the following properties:

(i) $p_t f(x)$ is differentiable with respect to t and twice Frechet differentiable with respect to x in directions of H, with trace class second Frechet derivative.

(ii) $(\partial/\partial t) p_t f(x) = \frac{1}{2} \operatorname{trace}[(p_t f)''(x)]$.

If f is not Lip 1, $(p_t f)''(x)$ need not be trace class.

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In this work we consider a second order differential operator of the form $\operatorname{trace}[A(x)f''(x)]$, where $A(x) = I - B(x)$ and $B(x)$ is trace class, acting on the space of bounded Lip 1 functions f on B. A set of measures $q_t(x, dy)$ $(0 < t < \infty, x \in B)$ are defined such that if $q_tf(x) = \int_B f(y)q_t(x, dy)$ for f a bounded Lip 1 function on B, then $q_tf(x)$ satisfies properties analogous to (i), (ii) and moreover $\lim_{t\to 0} q_tf(x) = f(x)$ uniformly in x. The method used for developing this "fundamental solution" $q_t(x, dy)$ yields the above regularity properties immediately, as well as a representation for $q_t(x, dy)$ which could be used to obtain further detailed information about the family $q_t(x, dy)$.

Daleckii [1] has obtained solutions to the Cauchy problem for equations of the form

$$\frac{\partial}{\partial t} F(x, t) = \operatorname{trace}[C(x, t)\, D^2F(x, t)]$$

$$F(x, 0) = \Phi(x)$$

where $C(x, t)$ is positive definite, and D^2 refers to the second Frechet derivative of F with respect to x in directions of B. However, in this case B is restricted to be a Hilbert space (a condition which fails in the case where B is Wiener space), and $\Phi(x)$ is required to be second Frechet differentiable in directions of B.

II. Preliminaries

This section covers the basic definitions and ideas necessary to the following work. Most of the material in this section can be found in Gross [8], [9].

Let L be a locally convex real linear space, and L^* its topological dual space. A weak distribution on L is defined to be an equivalence class of linear maps F from L^* to the space of real-valued random variables over a probability space (Ω, p) (depending on F). Two such maps F_1, F_2 are equivalent if, for any finite set of vectors $y_1, \ldots, y_n$ in L^*, the joint distribution of $F_i(y_1), \ldots, F_i(y_n)$ in R^n is the same for $i = 1$ or 2. A cylinder set in L is a set S of the form

$$S = \{x \in L / (\langle y_1, x\rangle, \ldots, \langle y_n, x\rangle) \in B\}$$

where $y_1, \ldots, y_n$ are in L^* and B is a Borel set in R^n. If K is any finite dimensional subspace of L^* containing $y_1, \ldots, y_n$ then S is said to be based on K. Let $\mathscr{R}$ be the ring of cylinder sets in L, and $\mathscr{S}_K$ the

σ-ring of cylinder sets based on a fixed K. A cylinder set measure μ on L is a real-valued non-negative finitely additive set function on $\mathscr{R}$ which is countably additive on each $\mathscr{S}_K$ and satisfies $\mu(L) = 1$. Given a weak distribution F on L, we can define an associated cylinder set measure with the property that, for any finite subset $y_1, ..., y_n$ of L^*, the random variables $F(y_1), ..., F(y_n)$ on Ω have the same joint distribution as $y_1, ..., y_n$ have when considered as random variables over $(L, \mathscr{R}, \mu)$. The converse of this assertion may also be established; hence the two concepts—weak distribution and cylinder set measure—are equivalent.

Let H be a real separable Hilbert space with norm $|\cdot|$, and identify H with H^* in the usual way. The particular weak distribution with which we will be concerned is the canonical normal distribution on H with variance parameter t. This distribution, n_t, is uniquely determined by the condition that for any x in H, $n_t(x)$ is normally distributed with mean zero and variance $t|x|^2$. It follows that n_t takes orthogonal vectors in H into independent random variables. A cylinder set S in H can be written in the form $S = P^{-1}(E)$, where P is a finite dimensional projection on H and E is a Borel set in the range of P. The cylinder set measure associated with n_t is called Gauss measure, and is given on S by

$$\nu_t(S) = (2\pi t)^{-n/2} \int_E \exp[-|x|^2/2t]\, dx$$

where n is the dimension of the range of P.

A tame function on H is a function of the form $f(x) = \phi(Px)$, where P is a finite dimensional projection on H and ϕ is a complex-valued Baire function on the range of P. If F is a representative of n_t, and if $y_1, ..., y_n$ is an orthonormal basis for the range of P, then $\tilde{f} \equiv \phi(F(y_1), ..., F(y_n))$ is a random variable on Ω. $\tilde{f}$ depends only on f and on F, and is independent of the particular representation of f in terms of ϕ and $y_1, ..., y_n$. Moreover, the distribution of $\tilde{f}$, the integrability of $\tilde{f}$, and the convergence in probability or in L^p of sequences $\{\tilde{f}_n\}$ are easily seen to depend only on n_t, and not on the choice of its representative F.

The map $f \rightarrow \tilde{f}$ from the algebra of tame functions on H to the algebra of random variables on Ω is a homomorphism, and becomes an isomorphism on restricting f to the subalgebra of continuous tame functions. This isomorphism may be extended to a larger class of continuous functions on H in a manner which may be found in Gross [6], and which we shall now describe.

A semi-norm $\|\cdot\|$ on H is called a measurable semi-norm if for each

$\epsilon > 0$ there exists a finite dimensional projection P_0 such that for the tame function $\| Px \|$ we have

$$\operatorname{prob}(\| Px \|^{\sim} > \epsilon) < \epsilon$$

whenever P is a finite dimensional projection orthogonal to P_0. Here prob $(\tilde{f} > \epsilon)$ refers to the distribution of the random variable $\tilde{f}$. It follows that a positive multiple of a measurable semi-norm is a measurable semi-norm, as is also the sum of two measurable semi-norms. If A is a non-negative trace class operator on H, then $\| x \|_A \equiv (Ax, x)^{\frac{1}{2}}$ is easily seen to be a measurable semi-norm.

Theorem 1 of [6] states that for a measurable semi-norm $\| \cdot \|$, the net $\| Px \|^{\sim}$ of tame functions converges in probability as P approaches I through the directed set $\mathscr{F}$ of finite dimensional projections on H. We call the limit random variable $\| x \|^{\sim}$.

Let H_m denote H with the topology determined by the measurable semi-norms. A function f on H is called uniformly continuous near zero in H_m (u.c.n. 0 in H_m) if there exists a sequence $\| \cdot \|_n$ of measurable semi-norms such that $\| \cdot \|_n^{\sim}$ converges to zero in probability while f is uniformly continuous with respect to the topology of H_m on the unit sphere of each $\| \cdot \|_n$.

Theorem 2 of [6] states that, for a complex-valued function f which is u.c.n. 0 in H_m, the net $f(Px)^{\sim}$ of tame functions converges in probability as P approaches I through $\mathscr{F}$. If $\tilde{f}$ denotes the limit random variable, then by Corollary 5.5 of [6], $\tilde{f} = 0$ a.e. if and only if f is identically equal to zero on H.

We will further restrict our attention to a particular probability space (Ω, p_t) associated with n_t. This space is obtained in the manner described below.

Henceforth $\| \cdot \|$ will denote a particular measurable norm on H. It follows from the definition of measurable norm that $\| x \| \leqslant \text{const.} | x |$ for all x in H. Let B be the completion of H with respect to $\| \cdot \|$. B is a real separable Banach space, and, moreover, any real separable Banach space can be obtained from H in this manner. Let i be the natural injection of H into B, and j the embedding (by restriction) of B^* into H^*. Note that i and j are continuous, and that B^* is dense in H^*, since B^* separates points of H. We will identify H with iH and B^* with jB^*. The triple (H, B, i) is called an abstract Wiener space.

Gauss measure ν_t on H induces a cylinder set measure ρ_t on B defined by

$$\rho_t\{x \in B/(\langle y_1, x\rangle, \ldots, \langle y_n, x\rangle) \in E\} = \nu_t\{x \in H/(\langle y_1, x\rangle, \ldots, \langle y_n, x\rangle) \in E\}$$

for all finite subsets $y_1, \ldots, y_n$ of B^* and Borel sets $E \subseteq R^n$. ρ_t is countably additive on the ring of tame sets of B, and thus has a unique countably additive extension p_t to the Borel field $\mathscr{S}$ of B. Moreover, the identity map on B^*, regarded as a densely defined map of H^* into random variables over $(B, \mathscr{S}, p_t)$ extends uniquely to a representative of n_t. p_t is called Wiener measure on B with variance parameter t. For $x \in B$, $t > 0$ we define $p_t(x, A) = p_t(A - x)$ for each A in $\mathscr{S}$. For $s, t > 0, p_s(x \cdot)$ and $p_t(y, \cdot)$ are equivalent measures if and only if $s = t$ and $x - y$ belongs to H. Otherwise they are mutually singular.

We will assume that $\| y \|^{\sim}$ is in $L^p(p_t(dy))$ for all $1 \leqslant p < \infty$ and for all $t > 0$. This does not appear to be a serious restriction, since it is satisfied by all of the measurable norms which are presently known.

Let W be a Banach space, whose norm we will denote by $| \cdot |_W$, omitting the subscript when it is clear that the W-norm is meant. If f is a W-valued function defined in a neighborhood of a point x of B, then the Frechet derivative of f at x is that (unique) element $y \in L(B, W)$ which satisfies $| f(x + x') - f(x) - y(x)' | = o(\| x' \|)$ for small $x' \in B$. ($L(B, W)$ is the space of bounded linear operators from B to W.) We will say that f is B-differentiable at x if such a y exists, and will write $y = Df(x)$. We may also regard f as a function g defined in a neighborhood of the origin of H by restricting f to the coset $x + H$ of B and defining $g(h) = f(x + h)$. The Frechet derivative of g at 0 is that element z of $L(H, W)$ which satisfies $| f(x + h) - f(x) - z(h) | = o(| h |)$ for small $h \in H$. We will say that f is H-differentiable at x if such a z exists, and will write $z = f'(x)$. Since the H-norm is stronger than the B-norm, $L(B, W)$ may be regarded as a subset of $L(H, W)$. Thus if f is B-differentiable at x then f is H-differentiable at x.

If $X \in L(H, H)$ then X is said to be trace class if trace $(X^*X)^{\frac{1}{2}} < \infty$. The trace class operators on H form a Banach space under the norm $| X |_{tr} = \text{trace}\,(X^*X)^{\frac{1}{2}}$. Viewing $H \subset B$ and $B^* \subset H^* = H$, the restriction of an operator Y in $L(B, B^*)$ to H gives rise to an element Y/H of $L(H, H)$. By Corollary 5 of [8] the symmetric part of Y/H is trace class. The subspace of $L(B, B^*)$ consisting of those Y such that Y/H is symmetric is closed, since the $L(H, H)$ norm of Y/H is dominated by $| Y/H |_{tr}$. It is a consequence of the closed graph theorem that $| Y/H |_{tr}$ is dominated by the $L(B, B^*)$ norm of Y.

If f is a real-valued function on B, then, letting $D^2f(x)$ and $f''(x)$ denote the second B- and H-derivatives respectively of f at x, we have $D^2f(x) \in L(B, B^*)$. Restricting to H, $D^2f(x)/H = f''(x)$, where $f''(x)$ is a symmetric member of $L(H, H)$ and therefore trace class.

However, if f is only twice H-differentiable at x, then the most we can say about $f''(x)$ is that it belongs to $L(H, H)$.

We will continue to use D and $'$ to denote B- and H-derivatives respectively, and will interpret $\| Df(x)\|$ and $|f'(x)|$ as the $L(B, W)$ and $L(H, W)$ norms respectively.

If f is a W-valued function defined in a neighborhood of a point x of B, then f will be said to be B-continuous at x if f is continuous at x in the norm topology for B. f will be called H-continuous at x if $g(h) \equiv f(x+h)$ $(h \in H)$ is continuous at 0 in the norm topology for H. It is obvious that B-continuity implies H-continuity. f will be called B-Lip α if there is a constant K such that $|f(x) - f(x')| \leqslant K\| x - x' \|^\alpha$ for all $x, x' \in B$. f will be called H-Lip α if there is a constant K such that $|f(x+h) - f(x)| \leqslant K| h |^\alpha$ for all $x \in B$, $h \in H$. Note that if f is B-Lip α then f is H-Lip α.

III. Background

The fundamental solution of the parabolic equation in R^n was developed by Feller [4] for $n = 1$, and this method was extended by Dressel ([2], [3]) to general n and by Itô [10] to a differentiable manifold. Yosida [12] considered the case of a Riemannian space, but by a different approach.

We follow the approach of Feller–Dressel–Itô, which is briefly sketched as follows:

Let

$$L_{x,t}u = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \, \partial x_j} - \frac{\partial u}{\partial t},$$

where $u = u(x, t)$, $a_{ij} = a_{ij}(x)$ $(x \in R^n,\ 0 < t \leqslant t_0)$. Under suitable smoothness conditions on $A(x)$ $(\equiv \{a_{ij}(x)\})$, we construct an initial approximation $Z(x, y, t)$ to the fundamental solution $\Gamma(x, y, t)$ of the equation $L_{x,t}u = 0$, given by

$$Z(x, y, t) = (4\pi t)^{-n/2} \exp[-(A^{-1}(x)(x-y), x-y)/4t] \cdot \det[A(y)]^{-1/2}. \quad (1)$$

We look for Γ in the form

$$\Gamma(x, y, t) = Z(x, y, t) + \int_0^t \int_{R^n} Z(x, z, t-s) f(z, y, s)\, dz\, ds \quad (2)$$

for a sufficiently smooth function f.

Since Γ is required to satisfy $L_{x,t}\Gamma = 0$, we apply $L_{x,t}$ to each side of (2), obtaining

$$0 = L_{x,t}Z + \int_0^t \int_{R^n} L_{x,t-s}Z(x, z, t - s) f(z, y, s)\, dz\, ds - f(x, y, t). \qquad (3)$$

This integral equation is then solved for f by iteration, and finally f is shown to satisfy the smoothness conditions necessary to the derivation of (3).

IV. Objective

Let $A(x) = I - B(x)$, where $B(\cdot)$ is a map from B to the space of symmetric trace class operators on H. If $f(x, t)$ is a real-valued measurable function on $B \times (0, \infty)$ we define

$$L_{x,t}f(x, t) = \operatorname{trace}[A(x)f''(x, t)] - \frac{\partial}{\partial t} f(x, t) \qquad (0 < t < \infty) \qquad (4)$$

whenever the right hand side exists. We will say that the right hand side exists when $\partial/\partial t(f(x, t)$ and $f''(x, t)$ exist and $[A(x)f''(x, t)]$ is trace class. When no danger of confusion arises, we will omit the subscripts on L. Under certain restrictions on the "coefficients" $A(x)$, which we will enumerate in the next section, we will construct a family of measures $q_t(x, dy)$ $(x \in B,\ 0 < t < \infty)$ on $(B, \mathscr{S})$ which satisfy

a-1) $L[\int_B f(y)q_t(x, dy)] = 0 \qquad (0 < t < \infty)$

a-2) $\lim_{t\to 0} \int_B f(y)q_t(x, dy) = f(x) \qquad$ (uniformly in x)

for all bounded B-Lip 1 functions f. We will call such a family of measures a fundamental solution of $Lf = 0$.

V. Hypotheses on the Coefficients

Since $A(x) = I - B(x)$, we will write our hypotheses in terms of $B(x)$. The following five conditions are sufficient, but by no means necessary, for the validity of our construction of $q_t(x, dy)$.

b-1) $x \to B(x)$ is a bounded B-Lip 1 function from B to symmetric trace class operators on H.

b-2) There exists $\epsilon > 0$ such that $B(x) \leqslant (1 - \epsilon)I$ for all $x \in B$.

b-3) There exists a symmetric Hilbert-Schmidt class operator E on H and a family of operators $B_0(x) \in L(H, H)$ such that for all $x \in B$, $B(x) = EB_0(x)E$ and $|B_0(x)| \leqslant 1$. (Here $|B_0(x)|$ refers to the operator norm of $B_0(x)$ as an element of $L(H, H)$.)

b-4) $D^2B_0(x)$ exists and is B-Lip 1 as a function of x.

b-5) $\| DB_0(x)\|$, $\| D^2B_0(x)\|$ are uniformly bounded for all $x \in B$.

Remarks (1) We note that b-1) can be deduced from b-3), b-4) and b-5).

(2) Instead of b-3) we could have imposed the seemingly weaker assumption that there exist symmetric Hilbert-Schmidt class operators E_1, E_2 on H such that $|(B(x)y_1, y_2)| \leqslant |E_1y_1| \cdot |E_2y_2|$ for all $y_1, y_2 \in H$ and for all $x \in B$. However, setting $E = (E_1^2 + E_2^2)^{\frac{1}{2}}$ and using the facts that $E_1^2 \leqslant E^2$, $E_2^2 \leqslant E^2$, we find that $B(x)$ satisfies $|(B(x)y_1, y_2)| \leqslant |Ey_1| \cdot |Ey_2|$. Now, by Lemma 4.2 of [5], there exists a (unique) family of operators $B_0(x) \in L(H, H)$ such that $|B_0(x)| \leqslant 1$, $B(x) = EB_0(x)E$, $\mathscr{R}[B_0(x)] \subset \mathscr{N}[E]^{\perp}$, and $\mathscr{N}[B_0(x)] \supset \mathscr{R}(F)^{\perp}$ where $\mathscr{N}[E]$ and $\mathscr{R}[F]$ denote the nullspace of E and range of F respectively.

Define $C(x) = [A(x)]^{-1} - I = [I - B(x)]^{-1}B(x)$. $C(x)$ is a symmetric trace class operator for each x, and

c-1) $x \to C(x)$ is a bounded B-Lip 1 function from B to trace class operators on H.

We will show that $C(x)$ satisfies

c-2) There exists a symmetric Hilbert-Schmidt class operator F on H and a family of operators $C_0(x) \in L(H, H)$ such that for all $x \in B$, $C(x) = FC_0(x)F$ and $|C_0(x)| \leqslant 1$.

The following derivation of c-2) is due to Leonard Gross. Let P be a finite dimensional projection on H and set $Q = I - P$.
Write $I - B(x) = I - PB(x)P - QB(x)P - PB(x)Q - QB(x)Q$. We can choose P to satisfy

(i) $PE = EP$

(ii) $|QB(x)P + PB(x)Q + QB(x)Q| \leqslant (1 - \epsilon)\epsilon$.

Factoring out $[I - PB(x)P]$, we obtain

$$I - B(x) = [I - PB(x)P][I - D(x)]$$

where

$$D(x) = [I - PB(x)P]^{-1}[QB(x)P + PB(x)Q + QB(x)Q].$$

$D(x)$ is symmetric and $|D| \leqslant 1 - \epsilon$. Moreover for all $y_1, y_2 \in H$ and $x \in B$

$$\begin{aligned}
&|(D(x)y_1, y_2)| \\
&\quad = |([I - PB(x)P]^{-1} E[QB_0(x)P + PB_0(x)Q + QB_0(x)Q]\, Ey_1, y_2)| \\
&\quad \leqslant 3\,|Ey_1| \cdot |E[I - PB(x)P]^{-1}y_2|.
\end{aligned}$$

To estimate the last term, we write

$$\begin{aligned}
&|E[I - PB(x)P]^{-1}y_2|^2 \\
&\quad = |E[I - PB(x)P]^{-1}Py_2 + E[I - PB(x)P]^{-1}Qy_2|^2 \\
&\quad = |E[I - PB(x)P]^{-1}Py_2|^2 + |EQy_2|^2 \\
&\quad \leqslant |E|^2\epsilon^{-2}|Py_2|^2 + |EQy_2|^2 \\
&\quad = |E_1y_2|^2,
\end{aligned}$$

say, where $E_1 = |E|\epsilon^{-1}P + EQ$. E_1 is a symmetric Hilbert-Schmidt class operator. It now follows from Lemma 4.2 of [5] that we may write $D(x) = E_1D_0(x)E_1$, where $|D_0(x)| \leqslant 1$. Then

$$\begin{aligned}
[I - D(x)]^{-1} &= I + \lim_{n\to\infty}\sum_{i=1}^{n}[D(x)]^i \\
&= I + E_1\left\{D_0(x) + \lim_{n\to\infty}\sum_{i=0}^{n} D_0(x)\,E_1[D(x)]^i\,E_1D_0(x)\right\}E_1 \\
&= I + E_1D_1(x)\,E_1, \quad \text{say, where} \quad |D_1(x)| \leqslant 1 + |E_1|^2\epsilon^{-1}.
\end{aligned}$$

Returning to $C(x)$, we have

$$\begin{aligned}
|(C(x)y_1, y_2)| &= |([I - D(x)]^{-1}[I - PB(x)P]^{-1}B(x)y_1, y_2)| \\
&\leqslant |(B_0(x)Ey_1, E[I - PB(x)P]^{-1}y_2)| \\
&\quad + |(D_1(x)E_1[I - PB(x)P]^{-1}EB_0(x)Ey_1, E_1y_2)| \\
&\leqslant |Ey_1| \cdot |E_1y_2| + [1 + |E_1|^2\epsilon^{-1}]|E_1|\epsilon^{-1}|E| \cdot |Ey_1| \cdot |E_1y_2| \\
&\leqslant |aEy_1| \cdot |E_1y_2|
\end{aligned}$$

where $a = 1 + [1 + |E_1|^2\epsilon^{-1}] \cdot |E_1|\epsilon^{-1}|E|$. c-2) now follows by the argument following b-5).

Using the formula

$$D([I - B(x)]^{-1})\, z = [I - B(x)]^{-1}\, DB(x)\, z[I - B(x)]^{-1},$$

we can show that $|(DC(x)zy_1, y_2)| \leqslant |Fy_1| \cdot |Fy_2| \cdot \| z \|$ and $|(D^2C(x)z_1z_2y_1, y_2)| \leqslant |Fy_1| \cdot |Fy_2| \cdot \| z_1 \| \cdot \| z_2 \|$ (for all $x, z, z_1, z_2 \in B$ and $y_1, y_2 \in H$). We will indicate the derivation for $DC(x)$; that for $D^2C(x)$ is analogous.

Writing $DC(x)z = [I - B(x)]^{-1}DB(x)z[I - B(x)]^{-1}$, we obtain

$$|(DC(x)\, zy_1, y_2)| \leqslant |DB_0(x)\, z| \cdot |E[I - B(x)]^{-1} y_1| \cdot |E[I - B(x)]^{-1} y_2| .$$

Now

$$\begin{aligned} E[I - B(x)]^{-1} &= E[I - D(x)]^{-1} [I - PB(x)\, P]^{-1} \\ &= E[I - PB(x)\, P]^{-1} + EE_1D_1(x)\, E_1[I - PB(x)\, P]^{-1} \end{aligned}$$

and $| E[I - PB(x)\, P]^{-1}y | \leqslant | E_1 y |^2$. It follows that

$$|(DC(x)\, zy_1, y_2)| \leqslant |F_1y_1| \cdot |F_1y_2| \cdot \| z \|,$$

where F_1 is a symmetric Hilbert-Schmidt class operator. Without loss of generality we may assume that $F_1 = F$.

The argument of Remark (2) following b-5) now shows that we may define operators $C_1(x, z)$, $C_2(x, z_1, z_2)$ belonging to $L(H, H)$ such that $DC(x)z = FC_1(x, z)F$ and $D^2C(x)\, z_1z_2 = FC_2(x, z_1, z_2)F$, where $|C_1(x, z)| \leqslant \| z \|$ and $|C_2(x, z_1, z_2)| \leqslant \| z_1 \| \cdot \| z_2 \|$ for all $x, z, z_1, z_2 \in B$. Since $C_1(x, z)$ and $C_2(x, z_1, z_2)$ are linear in z, z_1, and z_2, we may define $C_1(x)\, z = C_1(x, z)$ and $C_2(x)\, z_1z_2 = C_2(x, z_1, z_2)$ where $C_1(x) \in L(B \to L(H, H))$ and $C_2(x) \in L(B \to L(B \to L(H, H)))$. It is easy to see that the operator norms $\| C_1(x)\|$ and $\| C_2(x)\|$ are bounded by 1. Thus we have

c-3) There exist families of operators $C_1(x) \in L(B \to L(H, H))$ and $C_2(x) \in L(B \to L(B \to L(H, H)))$ such that for all $x, z, z_1, z_2 \in B$, $DC(x)\, z = FC_1(x)\, zF$ and $D^2C(x)\, z_1z_2 = FC_2(x)\, z_1z_2F$ where $\| C_1(x)\|, \| C_2(x)\| \leqslant 1$.

Note that $C_2(x)\, z_1z_2 = C_2(x)\, z_2z_1$ for all $x, z_1, z_2 \in B$. $C(x)$ and its derivatives also satisfy

c-4) There exists a constant c such that for all $x_1, x_2, z, z_1, z_2 \in B$ and $y_1, y_2 \in H$

$$|([C(x_1) - C(x_2)]\, y_1, y_2)| \leqslant c \| x_1 - x_2 \| \cdot |Fy_1| \cdot |Fy_2|$$

$$|([DC(x_1) - DC(x_2)]\, zy_1, y_2)| \leqslant c \| x_1 - x_2 \| \cdot \| z \| \cdot |Fy_1| \cdot |Fy_2|$$

$$|([D^2C(x_1) - D^2C(x_2)]\, z_1z_2y_1, y_2)| \leqslant c \| x_1 - x_2 \| \cdot \| z_1 \| \cdot \| z_2 \| \cdot |Fy_1| \cdot |Fy_2|$$

To establish the first inequality, we write $C(x) = [I - B(x)]^{-1} - I$. Then

$$\begin{aligned} C(x_1) - C(x_2) &= [I - B(x_1)]^{-1} - [I - B(x_2)]^{-1} \\ &= [I - B(x_1)]^{-1} [I - B(x_2)][I - B(x_2)]^{-1} \\ &\quad - [I - B(x_1)]^{-1} [I - B(x_1)][I - B(x_2)]^{-1} \\ &= [I - B(x_1)]^{-1} E[B_0(x_1) - B_0(x_2)] E[I - B(x_2)]^{-1} \end{aligned}$$

and it follows that

$$\begin{aligned} |([C(x_1) - C(x_2)] y_1, y_2)| &\leqslant | B_0(x_1) - B_0(x_2)| \cdot |Fy_1| \cdot |Fy_2| \\ &\leqslant c \| x_1 - x_2 \| \cdot |Fy_1| \cdot |Fy_2| . \end{aligned}$$

The remaining inequalities follow on writing

$$\begin{aligned} DC(x) z &= [I - B(x)]^{-1} DB(x) z[I - B(x)]^{-1} \\ D^2C(x) z_1 z_2 &= [DC(x) z_2][DB(x) z_1][I - B(x)]^{-1} \\ &\quad + [I - B(x)]^{-1} [D^2B(x) z_1 z_2][I - B(x)]^{-1} \\ &\quad + [I - B(x)]^{-1} [DB(x) z_1][DC(x) z_2] \end{aligned}$$

and noting that each of the terms on the right hand sides of these equalities is uniformly bounded in norm and satisfies a Lip 1 condition in the variable x.

Example Let B be the space of real continuous functions on $[0, 1]$ which vanish at zero (Wiener space) and let H be that subset of Wiener space consisting of the absolutely continuous functions in B which have square integrable first derivatives. The inner product on H is given by $(x, y) = \int_0^1 x'(t) y'(t) \, dt$, where $'$ denotes the first derivative with respect to t. B may be viewed as the completion of H with respect to the sup norm. Let $g(u, v, w)$ be a real-valued function defined for $u \in R^n$, $0 \leqslant v, w \leqslant 1$, satisfying

(i) $g(u, 1, w) = 0 \qquad (u \in R^n, \quad 0 \leqslant w \leqslant 1)$.

(ii) $g(u, v, w)$ is continuously differentiable as a function of v, for each u and w.

(iii) $g_v(u, v, w) \in L^2[(0, 1) \times (0, 1)]$ for each $u \in R^n$, with uniformly bounded L^2-norm.

(iv) $g_v(u, v, w)$ is twice differentiable with respect to u, the second u-derivative is Lip 1 as a function of u, and both u-derivatives have uniformly bounded $L^2[(0, 1) \times (0, 1)]$-norm.

(v) The integral operator $G(u)$ on $L^2(0, 1)$ given by

$$[G(u)] f(v) = \int_0^1 g_v(u, v, w) f(w)\, dw$$

is non-negative for each $u \in R^n$.

Let $t_1, \ldots, t_n$ be n specified points of $[0, 1]$. Define

$$[B(x)]\, y(t) = \int_0^t \int_0^1 g(x(t_1), \ldots, x(t_n), v, w)\, y(w)\, dw\, dv$$

for all $x, y \in B$. Identifying $L^2(0, 1)$ with H by the mapping $h \to \int_0^t h(v)\, dv$, we see that the action of $B(x)$ on $L^2(0, 1)$ is given by

$$[B(x)] f(t) = \frac{d}{dt} \left[\int_0^t \int_0^1 g(x(t_1), \ldots, x(t_n), v, w) \int_0^w f(z)\, dz\, dw\, dv\right]$$

$$= \int_0^1 g(x(t_1), \ldots, x(t_n), t, w) \int_0^w f(z)\, dz\, dw$$

$$= - \int_t^1 \int_0^1 g_v(x(t_1), \ldots, x(t_n), v, w) \int_0^w f(z)\, dz\, dw\, dv.$$

The operator $K : L^2(0, 1) \to L^2(0, 1)$ defined by $Kf(t) = \int_0^t f(w)\, dw$ differs from a symmetric operator by a unitary operator U— i.e. $(UK)^* = UK$. But $(UK)^* = K^* U^*$, where $K^* f(t) = \int_t^1 f(w)\, dw$. If $B_0(x) \equiv UG(x)^* U$ and $E \equiv UK$, we see that $B(x) = EB_0(x) E$. Upon normalization of $B_0(x)$, it follows from (i)–(v) that $B(x)$ satisfies b-1)–b-5).

VI. Statement of Theorem 1

Theorem 1. *Assume* $\|\cdot\|$ *is in* $L^p(p_1)$ *for all* $1 \leqslant p < \infty$, *and that* $A(x) = I - B(x)$ *where* $B(x)$ *satisfies* b-1)–b-5). *Then there exists a family of measures* $q_t(x, dy)$ $(0 < t < \infty,\ x \in B)$ *such that if* $q_t f(x) \equiv \int_B f(y)\, q_t(x, dy)$ *then for each bounded* B-Lip 1 *function* f

d-1) $q_t f(x)$ *is differentiable with respect to* t, *twice Frechet differentiable with respect to* x *in directions of* H, *and* $[A(x)(q_t f)''(x)]$ *is trace class.*

d-2) $(\partial/\partial t) q_t f(x) = \operatorname{trace}[A(x)(q_t f)''(x)]$.

d-3) $\lim_{t \to 0} q_t f(x) = f(x)$ *uniformly in* x.

VII. A Measurability Lemma

In the work that follows, we will often be concerned whether a function defined on H gives rise to a measurable function on B. Lemma 1 gives sufficient conditions for a function to be u.c.n. 0 in H_m.

Let $H^{(N)} = H \times H \times \cdots \times H$ (N times).

Lemma 1. *Let $f(y_1, ..., y_N) : H^{(N)} \to \mathbf{C}$ be linear in each y_i. If $|f(y_1, ..., y_N)| \leqslant c |y_1|_1 \cdot |y_2|_2 \cdot \cdots \cdot |y_N|_N$ where $|\cdot|_i$ is a measurable semi-norm on H ($i = 1, ..., N$), then $G(y) \equiv f(y, y, ..., y)$ is* u.c.n. 0 *in H_m.*

Proof. Define $\|y\|_n = n^{-1} \sum_{i=1}^{N} |y|_i$. Then $\|\cdot\|_n$ is a measurable semi-norm for each $n = 1, 2, ...,$ and $\|y\|_n^{\sim}$ converges to zero in probability (with respect to each $p_t(dy)$ ($t > 0$)). Let $S_n = \{y / \|y\|_n \leqslant 1\}$ It suffices to show that for each n, given $\epsilon > 0$, there exists $\delta(n, \epsilon)$ such that $y, z \in S_n$, $\|y - z\|_n < \delta(n, \epsilon) \Rightarrow |G(y) - G(z)| < \epsilon$. Now

$$
\begin{aligned}
|G(y) - G(z)| &= |f(y - z, y, ..., y) \\
&\quad + f(z, y - z, ..., y) + \cdots + f(z, z, ..., y - z)| \\
&\leqslant c[|y - z|_1 \cdot |y|_2 \cdot \cdots \cdot |y|_N \\
&\quad + |z|_1 \cdot |y - z|_2 \cdot \cdots \cdot |y|_N + \cdots \\
&\quad + |z|_1 \cdot |z|_2 \cdot \cdots \cdot |y - z|_N].
\end{aligned}
$$

Let $\delta(n, \epsilon) = \epsilon/(cNn^N)$. Then since $y \in S_n \Rightarrow |y|_i \leqslant n$ and $\|y - z\|_n < \delta(n, \epsilon) \Rightarrow |y - z|_i < n \cdot \delta(n, \epsilon)$, we conclude that

$$
\begin{aligned}
|G(y) - G(z)| &< c[n \cdot \delta(n, \epsilon) \cdot n^{N-1} + n \cdot n \cdot \delta(n, \epsilon) \cdot n^{N-2} + \cdots \\
&\quad + n^{N-1} \cdot n \cdot \delta(n, \epsilon)] \\
&= cNn^N \delta(n, \epsilon) \\
&= \epsilon.
\end{aligned}
$$

Fix $x \in B$, $k, h \in H$. Some examples of functions occurring in the next section which satisfy the hypotheses of Lemma 1 are

(i) $|(C(x) y_1, y_2)| \leqslant |C_0(x)| \cdot |Fy_1| \cdot |Fy_2|$.

(ii) $|(C'(x) hy_1, y_2)| \leqslant |C_1(x) h| \cdot |Fy_1| \cdot |Fy_2|$.

(iii) $|(C''(x) khy_1, y_2)| \leqslant |C_2(x) kh| \cdot |Fy_1| \cdot |Fy_2|$.

VIII. THE INITIAL MEASURES $M_t(x, dy)$

Consider the family of measures $m_t(x, dy)$ $(t > 0, x \in B)$ given by

$$m_t(x, dy) = \exp[-(C(x)(x-y), x-y)/4t]^{\sim} \, p_{2t}(x, dy) \tag{5}$$

where the exponential term is regarded as the Radon-Nikodym derivative of $m_t(x, \cdot)$ with respect to $p_{2t}(x, \cdot)$. We must check that for each x the exponential term defines a measurable function on B. Since a translate by $x \in B$ of a measurable function is measurable, it suffices to show that $\exp[-(C(x)\,y, y)/4t]$ gives rise to a measurable function, where the convergence of the associated tame functions in probability is with respect to $p_{2t}(dy)$. By Lemma 1 $(C(x)\,y, y)/4t$ is u.c.n.0 in H_m, and since exp composed with a measurable function is measurable the result follows. By making the change of variables $y \to x + y$ and diagonalizing $C(x)$, a straightforward calculation verifies that

$$\int_B m_t(x, dy) = \det[(I + C(x))^{-1/2}] = [\det A(x)]^{1/2}. \tag{6}$$

Thus the exponential term is in L^1 with respect to $p_{2t}(x, \cdot)$, and so defines a Radon-Nikodym derivative. Since $\| B(x)\|_{tr}$ is uniformly bounded, and $A(x) \geqslant \epsilon I$, $[\det A(x)]$ is bounded and bounded away from zero for all $x \in B$ (Lemma 4.1 of [5]). The proof of this lemma also shows that if $x \to C(x)$ is B-Lip 1 as a map from B to trace class operators on H, then the map $x \to [\det A(x)]$ is B-Lip 1.

REMARK. The following correction to the proof of Lemma 4.1 of [5] is necessary in order to see that $x \to [\det A(x)]$ is B-Lip 1. Replace that portion of the proof of Lemma 4.1 which is on p. 417 by the following;
From the identity

$$(I + tC)^{-1} - (I + tC_0)^{-1} = -t(I + tC_0)^{-1}\,(C - C_0)(I + tC)^{-1}$$

we obtain

$$\begin{aligned} &C(I + tC)^{-1} - C_0(I + tC_0)^{-1} \\ &\quad = (C - C_0)(I + tC)^{-1} - tC_0(I + tC_0)^{-1}\,(C - C_0)(I + tC)^{-1}. \end{aligned}$$

It follows that if C_0 and C are in a bounded subset S of $\mathfrak{A}_\alpha$ then

$$\| \log(I + C) - \log(I + C_0)\|_1 \leqslant \alpha^{-1} \| C - C_0 \|_1 + \alpha^{-2} \{\sup_{C_0 \in S} \| C_0 \|\} \| C - C_0 \|_1 .$$

This completes the proof of the lemma.

Let $\sup_{x \in B} | B(x)| \leqslant b$. Then

$$I + C(x) = [A(x)]^{-1} = [I - B(x)]^{-1} \geqslant I/(1 + b).$$

For $\lambda \geqslant 0$, $I + C(x) + \lambda C(x) \geqslant I/(1 + b) - \lambda \cdot \sup_{x \in B} | C(x)| \cdot I$. But $| C(x)| \leqslant | B(x)| \cdot |[I - B(x)]^{-1} | \leqslant b/\epsilon$, and so

$$I + (1 + \lambda) C(x) \geqslant [1/(1 + b) - \lambda b/\epsilon] I.$$

Since, for $\lambda < \epsilon/b(1 + b)$, $|[I + (1 + \lambda) C(x)]^{-1} |$ is uniformly bounded for all $x \in B$, it follows from (6) that

$$\exp[-(C(x)(x - y), x - y)/4t] \in L^{1+\lambda}(p_{2t}(x, \cdot)),$$

with uniformly bounded $L^{1+\lambda}$-norm.

For the following proposition, let K be a constant satisfying

e-1) $\| y \|, | Fy | \leqslant K| y |$ for all $y \in H$.

e-2) $|([C(x + y) - C(x)] y_1, y_2)| \leqslant K| y | \cdot | Fy_1 | \cdot | Fy_2 |$,

e-3) $|([C'(x + y) - C'(x)] y_1 y_2, y_3)| \leqslant K| y | \cdot | y_1 | \cdot | Fy_2 | \cdot | Fy_3 |$,

e-4) $|([C''(x + y) - C''(x)] y_1 y_2 y_3, y_4)| \leqslant K| y | \cdot | y_1 | \cdot | y_2 | \cdot |Fy_3| \cdot | Fy_4 |$

for all $x \in B$, $y, y_1, y_2, y_3, y_4 \in H$.

In the proof of this proposition, whenever $f(y)$ is a function defined for all $y \in H$ which is u.c.n.0 in H_m, we will write $\int_B f(y) p_t(dy)$ instead of $\int_B \tilde{f}(y) p_t(dy)$. In general, we will omit the $\sim$ whenever it is obvious that the corresponding random variable is intended. Define

$$(m_t f)(x) = \int_B f(y) m_t(x, dy) \tag{7}$$

for all bounded measurable f.

Proposition 1. *Let f be a bounded measurable function on B. Then for fixed $t > 0$, the function $x \to (m_t f)(x)$ is twice H-differentiable on B with first and second derivatives given by*

$$((m_t f)'(x), h) = (4t)^{-1} \int_B f(x + y) \cdot [-(C'(x) hy, y)$$

$$+ 2([A(x)]^{-1} h, y)] \cdot \exp[-(C(x) y, y)/4t] \cdot p_{2t}(dy) \tag{8}$$

$$\begin{aligned}((m_t f)''(x)\,k, h) = &-(4t)^{-1}\int_B f(x+y)\cdot\{[(C''(x)\,khy, y) - 2(C'(x)\,hk, y)\\ &- 2(C'(x)\,kh, y) + 2([A(x)]^{-1}h, k)] + [(C'(x)\,hy, y)\\ &- 2([A(x)]^{-1}h, y)]\cdot(4t)^{-1}\cdot[-(C'(x)\,ky, y)\\ &+ 2([A(x)]^{-1}k, y)]\}\cdot\exp[-(C(x)\,y, y)/4t]\cdot p_{2t}(dy) \qquad (9)\end{aligned}$$

where $h, k \in H$.

Proof. Let

$$\begin{aligned}G(x) &= (m_t f)(x)\\ &= \int_B f(x+y)\cdot\exp[-(C(x)\,y, y)/4t]\cdot p_{2t}(dy). \qquad (10)\end{aligned}$$

Since, by Theorem 3 of [*11*], $p_{2t}(x+h, dy)$ $(h \in H)$ is mutually absolutely continuous with respect to $p_{2t}(x, dy)$, it follows that for all $h \in H$

$$\begin{aligned}G(x+h) &= \int_B f(y)\cdot\exp[-(C(x+h)(x-y+h), x-y+h)/4t]\\ &\quad\cdot p_{2t}(x+h, dy)\\ &= \int_B f(x+y)\ \ J(h, y)\, p_{2t}(dy)\end{aligned}$$

where, for $y \in H$,

$$J(h, y) = \exp\{[2(h, y) - |\,h\,|^2 - (C(x+h)(y-h), y-h)]/4t\}. \quad (11)$$

Then, for $y \in H$,

$$\begin{aligned}\frac{\partial}{\partial s} J(sh, y) = (4t)^{-1}\,[2(h, y) - 2s\,|\,h\,|^2 &- (C'(x+sh)\,h(y-sh), y-sh)\\ &+ 2(C(x+sh)\,h, y-sh)]\cdot J(sh, y). \qquad (12)\end{aligned}$$

Thus

$$\begin{aligned}&\left|\, G(x+h) - G(x) - \int_B f(x+y)\cdot\frac{\partial}{\partial s} J(sh, y)_{/s=0}\cdot p_{2t}(dy)\right|\\ &\quad= \left|\int_B f(x+y)\left\{\int_0^1 \frac{\partial}{\partial s} J(sh, y)\, ds - \frac{\partial}{\partial s} J(sh, y)_{/s=0}\right\} p_{2t}(dy)\right|\\ &\quad\leqslant \|f\|_\infty \int_0^1\int_B \left|\frac{\partial}{\partial s} J(sh, y) - \frac{\partial}{\partial s} J(sh, y)_{/s=0}\right| p_{2t}(dy)\, ds.\end{aligned}$$

Let $\hat{K}$ represent a general constant, depending on $\|f\|_\infty$ and on t, and independent of h and x. We may restrict our consideration to $|h| \leqslant 1$. $\gamma_i(y)$ $(i = 1, 2, 3)$ will denote functions in $L^p(p_{2t}(\cdot))$ for all $1 \leqslant p < \infty$. An example of such a function is a polynomial in variables of the form $|y|_0^{\sim}$, where $|\cdot|_0$ is a measurable semi-norm given by a trace class operator on H. We make the following estimates:

$$\begin{aligned}
&\left| \frac{\partial}{\partial s} J(sh, y) - \frac{\partial}{\partial s} J(sh, y)_{/s=0} \right| \\
&\quad \leqslant (4t)^{-1} \mid -2s \mid h \mid^2 - (C'(x + sh)\, h(y - sh), y - sh) + (C'(x)\, hy, y) \\
&\qquad + 2(C(x + sh)\, h, y - sh) - 2(C(x)\, h, y)| \cdot J(sh, y) \ + (4t)^{-1} \mid 2(h, y) \\
&\qquad - (C'(x)\, hy, y) + 2(C(x)\, h, y)| \cdot | J(sh, y) - J(0, y)| \\
&\quad \leqslant \hat{K}[2s \mid h \mid^2 + K \mid sh \mid \cdot \mid h \mid \cdot \mid Fy \mid^2 + K^2 \mid h \mid \cdot (\mid Fy \mid + K \mid sh \mid) \cdot \mid sh \mid \\
&\qquad + K^2 \mid h \mid \cdot \mid sh \mid \cdot (\mid Fy \mid + K \mid sh \mid) + 2K \mid sh \mid \cdot \mid h \mid \cdot \mid Fy \mid \\
&\qquad + 2K^2 \mid h \mid \cdot \mid sh \mid]\ J(sh, y) + \hat{K}[2\, |(h, y)| + K \mid h \mid \cdot \mid Fy \mid^2 \\
&\qquad + 2K \mid h \mid \cdot \mid Fy \mid] \cdot | J(sh, y) - J(0, y)| \\
&\quad \leqslant \hat{K} \mid h \mid^2 \gamma_1(y) \cdot J(sh, y) + \hat{K} \mid h \mid \cdot \gamma_2(y) \cdot | J(sh, y) - J(0, y)| \,.
\end{aligned}$$

(Note that although $\gamma_2(y)$ involves a term of the form $|(h/|\,h\,|, y)|$, it does not depend on h, since

$$\int_B \left| \left(\frac{h}{|\,h\,|}, y \right) \right|^p p_{2t}(dy) = \int_{-\infty}^{\infty} |\,y\,|^p\, e^{-y^2/2t}\, dy$$

is independent of h.)

$$\begin{aligned}
&\int_B \gamma_1(y)\, J(sh, y)\, p_{2t}(dy) \\
&\quad = \int_B \gamma_1(y + sh) \cdot \exp[-(C(x + sh)\, y, y)/4t] \cdot p_{2t}(dy) \\
&\quad \leqslant \| \gamma_1(y + sh)\|_{(1+\lambda)/\lambda} \left\{ \int_B \exp[-((1 + \lambda)\, C(x + sh)\, y, y)/4t] \right. \\
&\qquad \left. \cdot\, p_{2t}(dy) \right\}^{1/(1+\lambda)} \\
&\quad = \| \gamma_1(y + sh)\|_{(1+\lambda)/\lambda} \{\det[I + (1 + \lambda)\, C(x + sh)]^{-1/2}\}^{1/(1+\lambda)} \leqslant \hat{K},
\end{aligned}$$

since

$$\begin{aligned}
&\|\gamma_1(y + sh)\|_{(1+\lambda)/\lambda} \\
&\quad = \left\{ \int_B \gamma_1(y)^{(1+\lambda)/\lambda} \cdot \exp[[2(y, sh) - |\,sh\,|^2]/4t] \cdot p_{2t}(dy) \right\}^{\lambda/(1+\lambda)} \\
&\quad \leqslant \| \gamma_1(y)\|_{2(1+\lambda)/\lambda} \left\{ \int_B \exp[[2(y, sh) - |\,sh\,|^2]/2t] \cdot p_{2t}(dy) \right\}^{\lambda/2(1+\lambda)} \\
&\quad \leqslant \hat{K} \qquad\qquad\qquad\qquad (0 \leqslant s \leqslant 1, \quad |\,h\,| \leqslant 1).
\end{aligned}$$

Similarly we find that

$$\begin{aligned}
&\int_B \gamma_2(y) \cdot | J(sh, y) - J(0, y)| \cdot p_{2t}(dy) \\
&\quad = \int_B \gamma_2(y) \cdot \left| \int_0^s \frac{\partial}{\partial r} J(rh, y)\, dr \right| \cdot p_{2t}(dy) \\
&\quad \leqslant \int_0^s \int_B \gamma_2(y) \cdot \left| \frac{\partial}{\partial r} J(rh, y) \right| \cdot p_{2t}(dy)\, dr \\
&\quad \leqslant \int_0^s \int_B \gamma_2(y) \cdot \hat{K} \,|\, h \,| \cdot \gamma_3(y) \cdot J(rh, y) \cdot p_{2t}(dy)\, dr \\
&\quad \leqslant \hat{K} |\, h \,|
\end{aligned}$$

We conclude that

$$\left| \frac{\partial}{\partial s} J(sh, y) - \frac{\partial}{\partial s} J(sh, y)_{/s=0} \right| = o(|\, h \,|).$$

Hence using the identity $I + C(x) = [A(x)]^{-1}$,

$$\begin{aligned}
(G'(x)\, h) &= \int_B f(x + y) \cdot \frac{\partial}{\partial s} J(sh, y)_{/s=0} \cdot p_{2t}(dy) \\
&= \text{right hand side of 8)} \\
&= -(4t)^{-1} \int_B f(y)[(C'(x)\, h(x - y), x - y) + 2([A(x)]^{-1}\, h, x - y)] \\
&\qquad \cdot \exp[-(C(x)(x - y), x - y)/4t] \cdot p_{2t}(x, dy).
\end{aligned}$$

For the second order derivative, we let $k \in H$ and restrict our attention to $|\, k \,| \leqslant 1$. When $\hat{K}$ is used, it will be independent of k. Then

$$\begin{aligned}
&(G'(x + k), h) \\
&\quad = -(4t)^{-1} \int_B f(y) \cdot [(C'(x + k)\, h(x - y + k), x - y + k) \\
&\qquad + 2([A(x + k)]^{-1}\, h, x - y + k)] \\
&\qquad \cdot \exp[-(C(x + k)(x - y + k), x - y + k)/4t] \cdot p_{2t}(x + k, dy) \\
&\quad = -(4t)^{-1} \int_B f(x + y) \cdot [(C'(x + k)\, h(y - k), y - k) \\
&\qquad - 2([A(x + k)]^{-1}\, h, y - k)] \\
&\qquad \cdot \exp\{[2(k, y) - |\, k \,|^2 - (C(x + k)(y - k), y - k)]/4t\} \cdot p_{2t}(dy).
\end{aligned}$$

For $y \in H$, define

$$J_1(k, y) = -(4t)^{-1} [(C'(x+k) h(y-k), y-k) - 2([A(x+k)]^{-1} h, y-k)]$$
$$J_2(k, y) = J_1(k, y)\, J(k, y).$$

Then

$$\begin{aligned}\frac{\partial}{\partial s} J_2(sk, y) = {} & -(4t)^{-1} [(C''(x+sk)\, kh(y-sk), y-sk) \\ & - 2(C'(x+sk)\, hk, y-sk) \\ & - 2(C'(x+sk)\, kh, y-sk) + 2([A(x+sk)]^{-1} h, k)] \\ & \cdot J(sk, y) + J_1(sk, y) \cdot \frac{\partial}{\partial s} J(sk, y).\end{aligned}$$

$$\begin{aligned}&\left|(G'(x+k), h) - (G'(x), h) - \int_B f(x+y) \cdot \frac{\partial}{\partial s} J_2(sk, y)_{/s+0} \cdot p_{2t}(dy)\right| \\
&\leqslant \| f \|_\infty \int_0^1 \int_B \left| \frac{\partial}{\partial s} J_2(sk, y) - \frac{\partial}{\partial s} J_2(sk, y)_{/s+0} \right| \cdot p_{2t}(dy) \cdot ds \\
&\leqslant (4t)^{-1} \| f \|_\infty \left\{ \int_0^1 \int_B |(C''(x+sk)\, kh(y-sk), y-sk) - (C''(x)\, kh, y) \right. \\
&\quad - 2(C'(x+sk)\, hk, y-sk) + 2(C'(x)\, hk, y) - 2(C'(x+sk)\, kh, y-sk) \\
&\quad + 2(C'(x)\, kh, y) + 2(C(x+sk)\, h, k) - 2(C(x)\, h, k)| \\
&\quad \cdot J(sk, y) \cdot p_{2t}(dy) \cdot ds \\
&\quad + \int_0^1 \int_B |(C''(x)\, khy, y) - 2(C'(x)\, hk, y) - 2(C'(x)\, kh, y) \\
&\quad + 2([A(x)]^{-1} h, k)| \cdot | J(sk, y) - J(0, y)| \cdot p_{2t}(dy) \cdot ds \\
&\quad + \int_0^1 \int_B |(C'(x+sk)\, h(y-sk), y-sk) - (C'(x)\, hy, y) \\
&\quad - 2([A(x+sk)]^{-1} h, y-sk) + 2([A(x)]^{-1} h, y)| \\
&\quad \cdot (4t)^{-1} \cdot | 2(k, y) - 2s\, | k |^2 \\
&\quad - (C'(x+sk)\, k(y-sk), y-sk) + 2(C(x+sk)\, k, y-sk)| \cdot J(sk, y) \\
&\quad \cdot p_{2t}(dy) \cdot ds + \int_0^1 \int_B |(C'(x)\, hy, y) - 2([A(x)]^{-1} h, y)| \\
&\quad \left. \cdot \left| \frac{\partial}{\partial s} J(sk, y) - \frac{\partial}{\partial s} J(sk, y)_{/s=0} \right| \right\} \\
&= O(|h|) \cdot o(| k |).\end{aligned}$$

Thus we obtain

$$(G''(x)\,k, h) = \int_B f(x+y) \cdot \frac{\partial}{\partial s} J_2(sk, y)_{/s=0} \cdot p_{2t}(dy)$$

and (9) follows immediately.

This concludes the proof of the proposition.

Remark. We note the following corrections to the statement and proof of the "if" part of Theorem 3 of [*11*]. Let the statement of Theorem 3 read:

For any closed densely-defined linear transformation T on the Hilbert space $\mathscr{H}$ with dense range, the transform by T of the canonical normal distribution n on $\mathscr{H}$ is absolutely continuous with respect to n if and only if $(T^*T)^{\frac{1}{2}}$ is nonsingular and has the form $I + B$, where B^2 is an operator with absolutely convergent trace. In the latter event the normal distribution with covariance operator $cA = cT^*T$ and general mean a in $\mathscr{H}$ is likewise absolutely continuous with respect to n, with derivative expressible in case B has absolutely convergent trace as

$$(\det A)^{-1/2} \exp[-\{((A^{-1} - I)\,x, x) - 2(A^{-1}x, a) + (A^{-1}a, a)\}/2c].$$

The proof of the "if" part of this theorem principally consists in demonstrating the convergence of the following infinite product:

$$\prod_{i=1}^{\infty} \int_{-\infty}^{\infty} \{[(2\pi\lambda_i c)^{-1/2} \exp(-(x - a_i)^2/2\lambda_i c)] \cdot [(2\pi c)^{-1/2} \exp(-x^2/2c)]^{-1}\}^{1/2} \cdot (2\pi c)^{-1/2} \exp(-x^2/2c)\, dx$$

In the work that follows we will use c to represent a general constant, which depends only on the coefficients $A(x)$.

Let $\mathfrak{A}$ be the Banach algebra of bounded B-Lip 1 functions on B, with norm given by

$$\|f\|_1 = \sup_{x\in B} |f(x)| + \inf\{c/|f(x) - f(y)| \leqslant c\,\|x - y\| \text{ for all } x, y \in B\}. \tag{13}$$

Define

$$\hat{f}(x) = [\det A(x)]^{-1/2} f(x) \quad \text{for all } f \in \mathfrak{A},\ x \in B. \tag{14}$$

PROPOSITION 2. *For $f \in \mathfrak{A}$, $m_t \hat{f}(x)$ is differentiable with respect to t and twice Frechet differentiable with respect to x in directions of H, with second Frechet derivative of trace class. The operators $(M_t f)(x) =$*

$L[m_t f(x)]$ $(0 < t < \infty,\ x \in B)$ are given by a family of measures $M_t(x, dy)$ which have the following properties:

f-1) *There is a constant Q, independent of t and x, such that $\int_B |M_t|(x, dy) \leqslant Qt^{-\frac{1}{2}}$.*

f-2) *The map $f \to M_t f$ is a bounded linear operator on $\mathfrak{A}$, with $\| M_t f \|_1 \leqslant Qt^{-\frac{1}{2}} \| f \|_1$.*

f-3) *Given $0 < \delta \leqslant t_0 < \infty$, there exists a constant Q_{δ, t_0}, depending on δ and t_0 but independent of f and x, such that for $\delta \leqslant t_1$, $t_2 \leqslant t_0$ we have $| M_{t_1} f(x) - M_{t_2} f(x)| \leqslant Q_{\delta, t_0} | t_1 - t_2 | \cdot \| f \|_1$ (for all $f \in \mathfrak{A}$).*

Remark. $M_t(x, dy)$ will correspond to the measure $Z(x, y, t) \cdot dy$ mentioned in Section III. δ may be arbitrarily close to zero, and t_0 arbitrarily large.

Proof. Since $x \to B(x)$ is bounded and B-Lip 1 as a map from B to trace class operators on H, and since $A(x) \geqslant \epsilon I$ for all $x \in B$, it follows from the proof of Lemma 4.1 of [5] that $[\det A(x)]^{-\frac{1}{2}}$ belongs to $\mathfrak{A}$. Thus $\hat{f} \in \mathfrak{A}$, and we have the estimate $\| \hat{f} \|_1 \leqslant c \| f \|_1$.

Fix $x_0 \in B$. Let $D(x_0) = [A(x_0)]^{-\frac{1}{2}}$. Define $\hat{H}$ as H with inner product $\{ , \}$ given by $\{a, b\} = (D(x_0)a, D(x_0)b)$. Rewrite $(m_t f)''(x_0) = V_t(x_0) + W_t(x_0)$, where V_t and W_t are given by

$$(V_t(x_0)\, k, h) = (2t)^{-1} \int_B f(x_0 + y) \cdot [-\{h, k\}\{k, y\}/2t] \cdot \exp[-(C(x_0)\, y, y)/4t] \cdot p_{2t}(dy)$$

$$\begin{aligned}(W_t(x_0)\, k, h) = \int_B f(x_0 + y) \cdot [&-(4t)^{-1} (C''(x_0)\, khy, y) \\ &+ (2t)^{-1} (C'(x_0)\, hk, y) + (2t)^{-1} (C'(x_0)\, kh, y) \\ &+ (16t^2)^{-1} (C'(x_0)\, hy, y) \cdot (C'(x_0)\, ky, y) - (8t^2)^{-1} \{h, y\} \\ &\cdot (C'(x_0)\, ky, y) - (8t^2)^{-1} \{k, y\} \cdot (C'(x_0)\, hy, y)] \\ &\cdot \exp[-(C(x_0)\, y, y)/4t] \cdot p_{2t}(dy).\end{aligned}$$

The $\hat{H}$ topology and the given topology for H are equivalent. Thus we may also regard B as the completion of $\hat{H}$ with respect to the semi-norm $\| \cdot \|$. We can now interpret

$$\hat{p}_{2t}(dy) \equiv \exp[-(C(x_0)\, y, y)/4t]^{\sim} \cdot p_{2t}(dy)$$

as the Wiener measure on B (with variance parameter $2t$) which is induced by Gauss measure on $\hat{H}$ (with variance parameter $2t$).

Then, defining $v(t, x_0) = \int_B f(x_0 + y)\, \hat{p}_{2t}(dy)$, we can compute

$$\{v''(t, x_0)\, k, h\} = (2t)^{-1} \int_B f(x_0 + y) \cdot [-\{h, k\} + \{h, y\}\{k, y\}/2t] \cdot \hat{p}_{2t}(dy) \tag{15}$$

where $'$ denotes differentiation in $\hat{H}$-directions (see Proposition 9 of [9]). By Theorem 3 of [9], for $f \in \mathfrak{A}$, $\partial v/\partial t$ exists, v'' exists and is trace class, and $\partial v/\partial t = \mathrm{trace}_{\hat{H}}[v''(t, x_0)]$, where the trace and second Frechet derivative are taken with respect to $\hat{H}$.

From the definition of V_t and the calculation of v'', we have $(V_t(x_0)\, k, h) = \{v''(t, x_0)\, k, h\}$ for all $k, h \in H$. Thus $V_t(x_0) = [A(x_0)]^{-1} v''(t, x_0)$, and so $v''(t, x_0) = A(x_0)\, V_t(x_0)$. Since $m_t f(x_0) = v(t, x_0)$ for all t, it follows that for $f \in \mathfrak{A}$ and for all $x \in B$, $\partial/\partial t[m_t f(x)]$ and $\mathrm{trace}([A(x)][V_t(x)])$ exist and

$$\frac{\partial}{\partial t}\, [m_t f(x)] = \mathrm{trace}([A(x)][V_t(x)]). \tag{16}$$

Now write $A(x_0) y = \sum_{j=1}^{\infty} \lambda_j (y, e_j) e_j$, where $\{e_j\}$ $(j = 1, 2,...)$ is an orthonormal basis for H. Note that $A(x_0) e_j = \lambda_j e_j$ is equivalent to $D(x_0) e_j = \lambda_j^{-1} e_j$. For $y \in H$, we will write y_j for (y, e_j). Then, proceeding formally, we interchange trace and integral to obtain

$$\begin{aligned}
&\mathrm{trace}[A(x_0)][W_t(x_0)] \\
&\quad = \int_B f(x_0 + y) \cdot \Big[\mathrm{trace}[A(x_0)][-(4t)^{-1} \\
&\qquad \cdot (C''(x_0)(\cdot)(\cdot)\, y, y) + t^{-1}(C'(x_0)(\cdot)(\cdot), y) + (16t^2)^{-1} \\
&\qquad \cdot (C'(x_0)(\cdot)\, y, y) \otimes (C'(x_0)(\cdot)\, y, y)] - (4t^2)^{-1} \sum_{j=1}^{\infty} \{\lambda_j e_j, y\} \\
&\qquad \cdot (C'(x_0)\, e_j y, y)\Big] \cdot \exp[-(C(x_0)\, y, y)/4t] \cdot p_{2t}(dy).
\end{aligned} \tag{17}$$

But

$$\begin{aligned}
&\sum_{j=1}^{\infty} \{\lambda_j e_j, y\} \cdot (C'(x_0)\, e_j y, y) \\
&\qquad = \sum_{j=1}^{\infty} (\lambda_j \cdot \lambda_j^{-1/2} e_j, \lambda_j^{-1/2} y, e_j)(C'(x_0)\, e_j y, y) \\
&\qquad = \sum_{j=1}^{\infty} (C'(x_0)(y, e_j)\, y, y) \\
&\qquad = (C'(x_0)\, yy, y).
\end{aligned} \tag{18}$$

Replacing f by $\hat{f}$ in the definitions of V_t and W_t, equations (16)–(18) lead to

$$\begin{aligned}
M_t f(x) &= \int_B \hat{f}(x+y) \cdot \{\text{trace}[A(x)][-(4t)^{-1}(C''(x)(\cdot)(\cdot)\,y, y) \\
&\quad + t^{-1}(C'(x)(\cdot)(\cdot), y) + (16t^2)^{-1}(C'(x)(\cdot)\,y, y) \otimes (C'(x)(\cdot)\,y, y)] \\
&\quad - (4t)^{-1}(C'(x)\,yy, y)\}^{\sim} \cdot \exp[-(C(x)\,y, y)/4t]^{\sim} \cdot p_{2t}(dy) \\
&= \int_B f(y) \cdot [\det A(y)]^{-1/2} \{\text{trace}[A(x)] \\
&\qquad \cdot [-(4t)^{-1}(C''(x)(\cdot)(\cdot)(x-y), x-y) \\
&\qquad - t^{-1}(C'(x)(\cdot)(\cdot), x-y) + (16t^2)^{-1}(C'(x)(\cdot)(x-y), x-y) \\
&\qquad \otimes (C'(x)(\cdot)(x-y), x-y)] \\
&\qquad + (4t)^{-1}(C'(x)(x-y)(x-y), x-y)\}^{\sim} \\
&\qquad\ \cdot \exp[-(C(x)(x-y), x-y)/4t]^{\sim} \cdot p_{2t}(x, dy)
\end{aligned} \tag{19}$$

In order to show that the operator M_t is given by a measure $M_t(x, dy)$ which is absolutely continuous with respect to $p_{2t}(x, dy)$, we must show that the candidate for its Radon-Nikodym derivative given in (19) is in $L^1(p_{2t}(x, \cdot\,))$. This calculation will also verify that Fubini's Theorem applies in the case where we interchanged trace and integral to obtain (17). Making the change of variables $(x-y)/(2t)^{\frac{1}{2}} \to y$, and using the facts that for $X, Y \in L(H, H)$, $|XY|_{tr} \leqslant |X| \cdot |Y|_{tr}$ when Y is trace class, and that both $|A(\,\cdot\,)|$ and $[\det A(\,\cdot\,)]^{\frac{1}{2}}$ are bounded, we see that

$$\begin{aligned}
\int_B |M_t|(x, dy) \leqslant ct^{-1/2} \int_B &\{|(C''(x)(\cdot)(\cdot)\,y, y)|_{tr} + |(C'(x)(\cdot)(\cdot), y)|_{tr} \\
&+ |(C'(x)(\cdot)\,y, y) \otimes (C'(x)(\cdot)\,y, y)|_{tr} \\
&+ |(C'(x)\,yy, y)|\}^{\sim} \\
&\ \cdot \exp[-(C(x)\,y, y)/2]^{\sim} \cdot p_1(dy).
\end{aligned}$$

We will show that the expression within $\{\ \}$ is dominated by a function which is in $L^p(p_1)$ for all $1 \leqslant p < \infty$. $(C''(x)(\,\cdot\,)(\,\cdot\,)\,y, y)$ is the restriction to H of $([D^2C](x)(\,\cdot\,)(\,\cdot\,)\,y, y)$, which is a symmetric member of $L(B, B^*)$. Thus $(C''(x)(\,\cdot\,)(\,\cdot\,)\,y, y)$ is trace class and $|(C''(x)(\,\cdot\,)(\,\cdot\,)\,y, y)|_{tr} \leqslant c\|(C_2(x)(\,\cdot\,)(\,\cdot\,)(Fy), Fy)\|_{L(B,B^*)} \leqslant c|Fy|^2$. $(C'(x)(\,\cdot\,)(\,\cdot\,), y) = (C_1(x)(\,\cdot\,)F(\,\cdot\,), Fy)$ is trace class if for some orthonormal basis $\{e_i\}_{i=1,2,\ldots}$ of H, $\sum_{i=1} |C_1(x)\,e_i(Fe_i)|^2 < \infty$. Let $\{e_i\}_{i=1,2,\ldots}$ be such that $Fy = \sum_{i=1}^{\infty} \lambda_i(y, e_i)e_i$. Then

$$\sum_{i=1}^{\infty} |C_1(x)\,e_i(Fe_i)|^2 = \sum_{i=1}^{\infty} |\lambda_i|^2\, |C_1(x)\,e_ie_i| \leqslant c \sum_{i=1}^{\infty} |\lambda_i|^2 < c$$

since F is a Hilbert-Schmidt class operator. Thus we have $|(C'(x)(\cdot)(\cdot), y)|_{tr} \leqslant c|Fy|$. Since $(C'(x)(\cdot)y, y)$ is a bounded linear operator on H, with norm $\leqslant c|Fy|^2$, it may be identified with an element $z \in H$. But $|z \otimes z|_{tr} = |z|^2$, and so

$$|(C'(x)(\cdot)y, y) \otimes (C'(x)(\cdot)y, y)|_{tr} \leqslant c\,|Fy|^4.$$

Finally, $|(C'(x)yy, y)| = |(C_1(x)y(Fy), Fy)| \leqslant \|y\| \cdot |Fy|^2$, and so we conclude that there exists a c such that

$$\int_B |M_t|(x, dy) \leqslant ct^{-1/2}. \tag{20}$$

We now show that M_t is a bounded linear operator on $\mathfrak{A}$. From (19) we obtain

$$M_t f(x) = \int_B f(x + \sqrt{2t}y) \cdot g(x, y, t)^\sim \cdot \exp[-(C(x)y, y)/2]^\sim \cdot p_1(dy) \tag{21}$$

where, for $x \in B$, $y \in H$, $0 < t \leqslant t_0$

$$\begin{aligned} g(x, y, t) = \operatorname{trace}[A(x)][-(\tfrac{1}{2})(C''(x)(\cdot)(\cdot)y, y) + (2/t)^{1/2} \\ \cdot (C'(x)(\cdot)(\cdot), y) + (\tfrac{1}{4}) \cdot (C'(x)(\cdot)y, y) \otimes (C'(x)(\cdot)y, y)] \\ - (2t)^{-1/2}(C'(x)yy, y). \end{aligned} \tag{22}$$

For $x_1, x_2 \in B$,

$$\begin{aligned} &|g(x_1, y, t) - g(x_2, y, t)| \\ &\quad \leqslant ct^{-1/2}\{|A(x_1) \cdot (C''(x_1)(\cdot)(\cdot)y, y) - A(x_2) \cdot (C''(x_2)(\cdot)(\cdot)y, y)|_{tr} \\ &\qquad + |A(x_1) \cdot (C'(x_1)(\cdot)(\cdot), y) - A(x_2) \cdot (C'(x_1)(\cdot)(\cdot), y)|_{tr} \\ &\qquad + |A(x_1) \cdot (C'(x_1)(\cdot)y, y) \otimes (C'(x_1)(\cdot)y, y) \\ &\qquad\qquad - A(x_2) \cdot (C'(x_2)(\cdot)y, y) \otimes (C'(x_2)(\cdot)y, y)|_{tr} \\ &\qquad + |(C'(x_1)yy, y) - (C'(x_2)yy, y)|\} \\ &\quad = ct^{-1/2}\{(i) + (ii) + (iii) + (iv)\}, \end{aligned}$$

say. Writing

$$\begin{aligned} (i) \quad &\leqslant |A(x_1) - A(x_2)| \cdot |(C''(x_1)(\cdot)(\cdot)y, y)|_{tr} \\ &\quad + |A(x_2)| \cdot |(C''(x_1)(\cdot)(\cdot)y, y) - (C''(x_2)(\cdot)(\cdot)y, y)|_{tr} \end{aligned}$$

and using the facts that $x \to A(x)$ is B-Lip 1 and uniformly bounded as a map from B to $L(H, H)$, $|(C''(x)(\cdot)(\cdot)y, y)|_{tr} \leqslant c|Fy|^2$, and

$$\begin{aligned} &|(C''(x_1)(\cdot)(\cdot)y, y) - (C''(x_2)(\cdot)(\cdot)y, y)| \\ &\quad \leqslant c\,\|([C_2(x_1) - C_2(x_2)](\cdot)(\cdot)(Fy), Fy)\|_{L(B,B^*)} \\ &\quad \leqslant c\,|Fy|^2\,\|x_1 - x_2\|, \end{aligned}$$

we obtain (i) $\leqslant c\gamma_4(y) \cdot \| x_1 - x_2 \|$ where $\gamma_4(y)^\sim \in L^p(p_1)$ for all $1 \leqslant p < \infty$. Making similar estimates on (ii), (iii), and (iv), we arrive at

$$| g(x_1, y, t) - g(x_2, y, t)| \leqslant ct^{-1/2}\gamma_5(y) \cdot \| x_1 - x_2 \| \tag{23}$$

where $\gamma_5(y)^\sim \in L^p(p_1)$ for all $1 \leqslant p < \infty$. We now write

$$\begin{aligned} | M_t f(x_1) - M_t f(x_2)| & \\ \leqslant \int_B & | \hat{f}(x_1 + \sqrt{2t}\, y) - \hat{f}(x_2 + \sqrt{2t}\, y)| \cdot | g(x_1, y, t)|^\sim \\ & \cdot \exp[-(C(x_1)\, y, y)/2]^\sim \cdot p_1(dy) \\ & + \int_B | \hat{f}(x_2 + \sqrt{2t}\, y)| \cdot | g(x_1, y, t) - g(x_2, y, t)|^\sim \\ & \cdot \exp[-(C(x_1)\, y, y)/2]^\sim \cdot p_1(dy) \\ & + \int_B | \hat{f}(x_2 + \sqrt{2t}\, y)| \cdot | g(x_2, y, t)|^\sim \\ & \cdot | \exp[-(C(x_1)\, y, y)/2] - \exp[-(C(x_2)\, y, y)/2]|^\sim \cdot p_1(dy) \end{aligned}$$

and use $\| \hat{f} \|_1 \leqslant c\| f \|_1$, (23), and

$$\begin{aligned} \int_B & | \exp[-(C(x_1)\, y, y)/2] - \exp[-(C(x_2)\, y, y)/2]|^{1+\lambda}\, p_1(dy) \\ & = \int_B \Big| -(\tfrac{1}{2}) \int_0^1 (DC(x_2 + s(x_1 - x_2))(x_1 - x_2)\, y, y) \\ & \qquad \cdot \exp[-(C(x_2 + s(x_1 - x_2))\, y, y)/2]\, ds \Big|^{1+\lambda} p_1(dy) \\ & \leqslant c \, \| x_1 - x_2 \| \quad \text{for} \quad \lambda < \epsilon/2 \end{aligned}$$

to obtain

$$| M_t f(x_1) - M_t f(x_2)| \leqslant ct^{-1/2} \| f \|_1 \cdot \| x_1 - x_2 \| . \tag{24}$$

From (13), (20), and (24) we conclude that there exists a constant Q, independent of t, such that

$$\| M_t f \|_1 \leqslant Q t^{-1/2} \| f \|_1 , \quad \text{and} \quad \int_B | M_t |(x, dy) \leqslant Q t^{-1/2}. \tag{25}$$

We also note that for fixed $f \in \mathfrak{A}$, $x \in B$, $M_t f(x)$ is a continuous function of t for $t > 0$, since, from (21),

$$\begin{aligned} | M_{t_1} f(x) - M_{t_2} f(x)| < \int_B & | \hat{f}(x + \sqrt{2t_1}\, y) - \hat{f}(x + \sqrt{2t_2}\, y)| \cdot | g(x, y, t_1)|^\sim \\ & \cdot \exp[-(C(x)\, y, y)/2]^\sim \cdot p_1(dy) + \int_B | \hat{f}(x + \sqrt{2t_2}\, y \,| \cdot | g(x, y, t_1) \\ & - g(x, y, t_2)|^\sim \cdot \exp[-(C(x)\, y, y)/2]^\sim \cdot p_1(dy) \end{aligned}$$

and since

$$|\hat{f}(x + \sqrt{2t_1}\, y) - \hat{f}(x + \sqrt{2t_2}\, y)| \leqslant \|f\|_1 \cdot | \sqrt{2t_1} - \sqrt{2t_2} | \cdot \| y |$$

and $| g(x, y, t_1) - g(x, y, t_2)| \leqslant c|(t_1)^{-\frac{1}{2}} - (t_2)^{-\frac{1}{2}} | \cdot \gamma_6(y)$, where $\gamma_6(y)^{\sim} \in L^p(p_1)$ for all $1 \leqslant p < \infty$. In fact, for $0 < \delta \leqslant t_1$, $t_2 \leqslant t_0$ there exists a constant Q_{δ, t_0} , depending on δ and t_0 , but independent of f and x, such that

$$| M_{t_1} f(x) - M_{t_2} f(x)| \leqslant Q_{\delta, t_0} \, | t_1 - t_2 | \cdot \|f\|_1 . \tag{26}$$

This concludes the proof of Proposition 2.

IX. The Integral Equation

As in the finite dimensional case, we look for a fundamental solution $q_t(x, dy)$ of the form

$$\begin{aligned} \int_B f(y)\, q_t(x, dy) = \int_B \hat{f}(y)\, m_t(x, dy) \\ + \int_0^t \int_B \Big\{ [\det A(y)]^{-1/2} \\ \cdot \int_B f(z) \cdot r_u(y, dz) \Big\}\, m_{t-u}(x, dy) \cdot du \end{aligned} \tag{27}$$

for all $f \in \mathcal{O}\!\ell$, where $r_t(x, dy)$ $(0 < t < \infty,\ x \in B)$ is a family of finite Borel measures on B which satisfy conditions similar to f-1)–f-3) with Q and Q_{δ, t_0} replaced by c_{t_0} and c_{δ, t_0} respectively. We further require that for $f \in \mathcal{O}\!\ell$,

$$L \Big\{ \int_B f(y)\, q_t(x, dy) \Big\} = 0.$$

An application of $\dot{L}_{x,t}$ to each side of (27) leads to

$$0 = M_t f(x) + \int_0^t M_{t-u}[r_u f](x) \cdot du - h(x, t),$$

where

$$h(x, t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_t^{t+\Delta t} \int_B \widehat{r_u f}(y) \cdot m_{t+\Delta t-u}(x, dy) \cdot du.$$

Making the change of variables $y \to x + \sqrt{2(t + \Delta t - u)}\, y$, and using Fubini's Theorem and the Dominated Convergence Theorem to interchange the integrals and limit, we have

$$h(x, t) = \int_B \left\{ \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_t^{t+\Delta t} \widehat{r_u f}(x + \sqrt{2(t + \Delta t - u)}\, y) \cdot du \right\}$$
$$\cdot \exp[-(C(x)\, y, y)/2] \cdot p_1(dy).$$

By f-2) and f-3) the expression within the inner integral is a continuous function of u and of Δt, for Δt sufficiently small. It now follows from the Fundamental Theorem of Calculus that

$$\begin{aligned} h(x, t) &= \int_B r_t f(x) \cdot [\det A(x)]^{-1/2} \cdot \exp[-(C(x)\, y, y)/2] \cdot p_1(dy) \\ &= r_t f(x). \end{aligned}$$

Thus, for each $f \in \mathfrak{A}$, $r_t f$ must satisfy the following integral equation

$$r_t f(x) = M_t f(x) + \int_0^t M_{t-u}[r_u f](x)\, du. \tag{28}$$

Proposition 3. *There exists a family of finite Borel measures $r_t(x, dy)$ $(0 < t < \infty,\ x \in B)$ on B which satisfy equation* (28) *and the following conditions*

g-1) *Given $t_0 > 0$, there is a constant c_{t_0}, independent of t and x, such that $\int_B | r_t |(x, dy) \leqslant c_{t_0} \cdot t^{-\frac{1}{2}}$ $\quad (0 < t \leqslant t_0)$.*

g-2) *The map $f \to r_t f$ defined by $r_t f(x) = \int_B f(y)\, r_t(x, dy)$ is a bounded linear operator on $\mathfrak{A}$, with $\| r_t f \|_1 \leqslant c_{t_0} \cdot t^{-\frac{1}{2}} \| f \|_1$ $(0 < t \leqslant t_0)$.*

g-3) *Given $0 < \delta \leqslant t_0$, there exists a constant c_{δ, t_0}, depending on δ and t_0 but independent of f and x, such that for $\delta \leqslant t_1$, $t_2 \leqslant t_0$, $| r_{t_1} f(x) - r_{t_2} f(x)| \leqslant c_{\delta, t_0} | t_1 - t_2 | \cdot \| f \|_1$ (for all $f \in \mathfrak{A}$).*

Proof. We will solve (28) by iteration. Define the measures $r_t^n(x, dy)$ $(n = 1, 2, \ldots,\ 0 < t < \infty,\ x \in B)$ by the conditions

$$\begin{aligned} &\int_B f(y)\, r_t^1(x, dy) = \int_B f(y)\, M_t(x, dy) \\ &\int_B f(y)\, r_t^n(x, dy) = \int_0^t \int_B \int_B f(z)\, r_u^{n-1}(y, dz)\, M_{t-u}(x, dy)\, du \quad (n = 2, 3, \ldots) \end{aligned} \tag{29}$$

for all bounded measurable functions f on B. By (25), $r_t^n(x, dy)$ satisfies

$$\int_B |r_t^1|(x, dy) \leqslant Qt^{-1/2}$$

$$\int_B |r_t^2|(x, dy) \leqslant Q^2 \int_0^t (t-u_1)^{-1/2} u_1^{-1/2} \, du_1$$

$$\int_B |r_t^n|(x, dy) \leqslant Q^n \int_0^t (t-u_{n-1})^{-1/2} \int_0^{u_{n-1}} (u_{n-1}-u_{n-2})^{-1/2} \cdots$$
$$\cdot \int_0^{u_2} (u_2-u_1)^{-1/2} u_1^{-1/2} \, du_1 \cdots du_{n-1} \; (n = 3, 4, \ldots).$$

Making use of the identity

$$\int_0^a (a-u)^{b-1} u^{c-1} \, du = a^{b+c-1} \Gamma(b)\, \Gamma(c)/\Gamma(b+c)$$

where Γ is the gamma function and $a, b, c > 0$, we obtain

$$\begin{aligned}
\int_B |r_t^2|(x, dy) &\leqslant Q^2 t^0 \Gamma(\tfrac{1}{2})\, \Gamma(\tfrac{1}{2})/\Gamma(2/2) \\
&= Q^2 \pi^{2/2} t^{2/2-1}/\Gamma(2/2) \\
\int_B |r_t^3|(x, dy) &\leqslant [Q^3 \pi^{2/2}/\Gamma(2/2)] \int_0^t (t-u_2)^{-1/2} u_2^{2/2-1} \, du_2 \\
&= [Q^3 \pi^{2/2}/\Gamma(2/2)]\, t^{3/2-1}\, \Gamma(\tfrac{1}{2})\, \Gamma(2/2)/\Gamma(3/2) \\
&= Q^3 \pi^{3/2} t^{3/2-1}/\Gamma(3/2) \\
\int_B |r_t^n|(x, dy) &\leqslant Q^n \pi^{n/2} t^{n/2-1}/\Gamma(n/2).
\end{aligned}$$

Thus the series $\sum_{n=1}^{\infty} \int_B |r_t^n|(x, dy)$ converges, and for $0 < t \leqslant t_0$ its sum is dominated by a term of the form $c_{t_0} t^{-\frac{1}{2}}$. This enables us to define measures $r_t(x, dy)$ by

$$r_t(x, dy) = \sum_{n=1}^{\infty} r_t^n(x, dy). \tag{30}$$

The family $r_t(x, dy)$ satisfies condition g-1) and equation (28). To see that g-2) is satisfied, we let $f \in \mathfrak{A}$ and write

$$\begin{aligned}
r_t^1 f(x) &= M_t f(x) \\
r_t^n f(x) &= \int_0^t M_{t-u}[r_u^{n-1} f](x) \, du \\
r_t f(x) &= \sum_{n=1}^{\infty} r_t^n f(x).
\end{aligned}$$

Using (25) we can obtain the estimate

$$\| r_t^n f \|_1 \leqslant Q^n \| f_1 \| \pi^{n/2} t^{(n-2)/2} / \Gamma(n/2), \tag{31}$$

and g-2) follows immediately.
Let $0 < \delta \leqslant t_1, t_2 \leqslant t_0$. Then, by (25), (26) and (31)

$$\begin{aligned} | r_{t_1}^n f(x) - r_{t_2}^n f(x) | &\leqslant \left| \int_{t_1}^{t_2} M_{t_2-u}[r_u^{n-1} f](x)\, du \right| \\ &\quad + \left| \int_0^{t_1} \{ M_{t_2-u}[r_u^{n-1} f](x) - M_{t_1-u}[r_u^{n-1} f](x) \}\, du \right| \\ &\leqslant \int_{t_1}^{t_2} Q(t_2 - u)^{-1/2} \| r_u^{n-1} f \|_1 \\ &\quad + \int_0^{t_1} Q_{\delta, t_0} | t_2 - t_1 | \cdot \| r_u^{n-1} f \|_1\, du \\ &\leqslant c_{\delta, t_0} | t_2 - t_1 | \cdot \| f \|_1 , \end{aligned}$$

and so we have g-3) satisfied.

X. Proof of Theorem 1

Define $q_t(x, dy)$ $(0 < t < \infty, x \in B)$ by

$$\begin{aligned} \int_B f(y)\, q_t(x, dy) &= \int_B f(y)\, [\det A(y)]^{-1/2}\, m_t(x, dy) \\ &\quad + \int_0^t \int_B [\det A(y)]^{-1/2} \int_B f(z)\, r_u(y, dz)\, m_{t-u}(x, dy)\, du \end{aligned} \tag{32}$$

for all bounded measurable functions f on B.

It remains only to verify d-3). Writing

$$f(x) = \int_B f(x)\, m_t(x, dy),$$

we find that for t sufficiently small, say $0 < t \leqslant 1$,

$$\begin{aligned} &\left| \int_B f(y) \cdot q_t(x, dy) - f(x) \right| \\ &\quad \leqslant \int_B | f(y) - f(x) | \cdot m_t(x, dy) + c \| f \|_1 \int_0^t u^{-1/2}\, du \\ &\quad \leqslant c \| f \|_1 \int_B \| y - x \| \cdot m_t(x, dy) + c \| f \|_1 t^{1/2}. \end{aligned}$$

Make the change of variable $y \to x - \sqrt{2t}\, y$ in the first term. Then

$$\left|\int_B f(y) \cdot q_t(x, dy) - f(x)\right| \leqslant c \, \|f\|_1 \, t^{1/2} \int_B \|y\| \cdot m_1(dy) + c \, \|f_1\| \, t^{1/2}$$

$\to 0$ as $t \to 0$, and the convergence is uniform in x.

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